SOME GENERALIZATIONS OF PALEY'S THEOREMS ON FOURIER SERIES WITH POSITIVE COEFFICIENTS*

BY

MICHAEL FEKETE

1. **Introduction.** Let $f(x)$ be a real-valued function of a real variable x, periodic with the period 2π and Lebesgue integrable. These properties will be assumed throughout, without being explicitly stated. Let, in addition, for all x,

$$|f(x)| \leqq L < \infty. \tag{1.1}$$

Let

$$f(x) \sim a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1.2}$$

be the Fourier series of $f(x)$,

$$s_0(x) = a_0,\ s_n(x) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) \quad (n = 1, 2, 3, \cdots) \tag{1.3}$$

the partial sums of (1.2), and

$$\sigma_n(x) = a_0 + \sum_{\nu=1}^{n} \left(1 - \frac{\nu}{n}\right)(a_\nu \cos \nu x + b_\nu \sin \nu x) \quad (n = 1, 2, 3, \cdots) \tag{1.4}$$

their arithmetic means.

It is a classical result of the theory of Fejér and Lebesgue that the sequence $\{\sigma_n(x)\}$ is uniformly bounded and satisfies

$$|\sigma_n(x)| \leqq L, \tag{1.5}$$

and that, as $n\to\infty$, $\sigma_n(x)\to f(x)$ uniformly over $(-\pi, \pi)$ provided $f(x)$ is continuous in $(-\pi, \pi)$. As to the partial sums (1.3) themselves, as may be shown by suitable examples, they need not be uniformly bounded even when (1.1) is satisfied, and the sequence $\{s_n(x)\}$ need not converge uniformly to $f(x)$ even when $f(x)$ is continuous in $(-\pi, \pi)$.

Under these circumstances special attention should be given to a recent result of Paley† according to which the non-negativeness of the Fourier coeffi-

* Presented to the Society, February 23, 1935; received by the editors April 17, 1934.

† R. E. A. C. Paley, *On Fourier series with positive coefficients*, Journal of the London Mathematical Society, vol. 7 (1932), pp. 205–208. On the basis of (1.1) Paley derives the estimate $|s_n(x)| \leqq 10L$.

cients a_n, b_n of $f(x)$ combined with (1.1) implies the uniform boundedness of $\{s_n(x)\}$, while combined with the continuity of $f(x)$, it implies the uniform convergence of $s_n(x)$ to $f(x)$.

In a letter to Professor Fejér, written in the autumn of 1932,* Paley stated and gave a sketch of a proof of the fact that the same results hold if the condition of non-negativeness of a_n, b_n is replaced by a less restrictive one, viz.,

$$(1.6) \qquad a_n \geqq -K/n, \qquad b_n \geqq -K/n, \qquad 0 \leqq K < \infty.\dagger$$

After learning of these latter results of Paley's, the author of the present paper completed his proof with various improvements in the estimates‡ and also succeeded in extending these results to the generalized Fourier series of almost periodic functions of H. Bohr, $\sum(a_n \cos \lambda_n x + b_n \sin \lambda_n x)$. These investigations§ of the generalized series suggested, in the case of the ordinary series, the replacement of (1.6) by conditions

$$(1.7) \qquad a_n + \alpha_n \geqq 0, \qquad b_n + \beta_n \geqq 0 \qquad (n = 1, 2, 3, \cdots),$$

where

$$(1.8) \qquad \alpha_n \geqq 0, \qquad \beta_n \geqq 0,$$

while the series $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$ are "slowly divergent" in the sense of one of the definitions which follow.

DEFINITION 1′. *A series $\sum_1^\infty c_n$ with non-negative terms is said to be (at most) slowly divergent if there exist two positive numbers P and p, and a positive integer N, such that*

$$\sum_{\nu=n}^{n+q} c_\nu \leqq P, \quad \textit{for } n \geqq N, \quad q \leqq pn.$$

DEFINITION 1″. *A series $\sum_1^\infty c_n$ with non-negative terms is said to be (at most) slowly divergent if for an arbitrarily given positive P there exist a positive number p and a positive integer N, both depending on P, such that*

* This letter is reproduced in a note by Fejér, *On a theorem of Paley*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 469–475, especially pp. 474–475. On the basis of (1.1) and (1.6) Paley derives the estimate $|s_n(x)| \leqq K\epsilon + M_\epsilon L$ where ϵ is an arbitrary positive number while M_ϵ is a positive number which depends on ϵ but not on K and L.

† Analogous results have been found independently by Szász, *Zur Konvergenztheorie der Fourierschen Reihen*, Acta Mathematica, vol. 61 (1933), pp. 185–201.

‡ M. Fekete, *Proof of three propositions of Paley*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 138–144.

§ The previous results of Paley on Fourier series (of purely periodic functions) with positive coefficients have been already extended by the author in his paper *On generalized Fourier series with non-negative coefficients*, presented to the London Mathematical Society on November 16, 1933, forthcoming in their Proceedings.

$$\sum_{\nu=n}^{n+q} c_\nu \leqq P \text{ for } n \geqq N,\quad q \leqq pn.*$$

Since the harmonic series $\sum 1/n$ is slowly divergent in the sense of both Definitions 1′ and 1″ we thus obtain in (1.7) a generalization of conditions (1.6). It will be shown (Theorems 1 and 2 below) that if conditions (1.7), (1.8) are satisfied, the boundedness of $f(x)$ combined with the slow divergence of the series $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$ in the sense of Definition 1′ implies the uniform boundedness of the sequence $\{s_n(x)\}$, while the continuity of $f(x)$ together with the slow divergence of $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$ in the sense of Definition 1″ implies the uniform convergence to $f(x)$ of the sequence $\{s_n(x)\}$.

These results can be established in the same fashion as in the special case $\alpha_n = K/n$, $\beta_n = K/n$, but the method applied here has proved, after a slight modification, to be adequate to cope with a more general situation as far as the coefficients a_n, b_n are concerned. The modified, more general, conditions on a_n, b_n are suggested by the fact that conditions (1.8) together with the slow divergence of $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$ in the sense of Definition 1′ implies the "slow oscillation" of the series $\sum_1^\infty \alpha_n \cos nx$, $\sum_1^\infty \beta_n \sin nx$, uniformly in x, in the sense of Definition 2′ below and the Remark appended, while (1.8) together with the slow divergence in the sense of Definition 1″ implies the slow oscillation of $\sum_1^\infty \alpha_n \cos nx$, $\sum_1^\infty \beta_n \sin nx$, uniformly in x, in the sense of Definition 2″ below and the Remark.

DEFINITION 2′. *A series $\sum_1^\infty c_n$ with real terms is said to be slowly oscillating if there exist two positive numbers P and p and a positive integer N such that*

$$\left|\sum_{\nu=n}^{n+q} c_\nu\right| \leqq P, \quad \text{for } n \geqq N,\quad q \leqq pn.$$

DEFINITION 2″. *A series $\sum_1^\infty c_n$ with real terms is said to be slowly oscillating if for an arbitrarily given positive P there exist a positive number p and a positive integer N, both depending on P, such that*

$$\left|\sum_{\nu=n}^{n+q} c_\nu\right| \leqq P, \quad \text{for } n \geqq N,\quad q \leqq pn.$$

Remark. If the terms c_n of the series $\sum_1^\infty c_n$ depend on a parameter t which ranges over an interval $c \leqq t \leqq d$, we shall say that the slow oscillation of $\sum_1^\infty c_n(t)$ is uniform in t over (c, d) if the series oscillates slowly (in the sense

* It is clear that slow divergence in the sense of Definition 1″ implies that of Definition 1′, but the converse is not true even if the general term of the series should tend to zero, as may be shown by examples. Incidentally the property required in Definition 1′ is equivalent to

$$\sum_{\nu=n}^{2n} c_\nu \leqq K < \infty \text{ for all } n.$$

of either Definition 2′ or 2″), the characteristic data of the slow oscillation being independent of t.

This raises the question whether the results above concerning the sequence $\{s_n(x)\}$ still hold if, without changing the hypotheses on $f(x)$, and retaining (1.7), we replace conditions (1.8) together with the slow divergence of $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$ by the condition of the uniform slow oscillation over $(-\pi, \pi)$ of the series $\sum_1^\infty \alpha_n \cos nx$, $\sum_1^\infty \beta_n \sin nx$ in the sense of Definitions 2′ or 2″. That this question can be answered in the affirmative is shown by Theorems 5 and 6 below. The modified conditions deserve special attention, for they lead to conditions which are not only sufficient but also necessary for the behavior under consideration of the sequence of partial sums $\{s_n(x)\}$ (Theorems 7 and 8). On the other hand, while conditions (1.8) together with the slow divergence of $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$ bear only upon the negative ones among the Fourier coefficients a_n, b_n, our modified conditions are of a more complicated nature and involve all the coefficients a_n, b_n, negative as well as positive.

In concluding this introduction the author wishes to state that he owes the notion of slow divergence to Professor Fejér, who defined and used the notion of slow oscillation in the sense of Definition 2″ in his investigations on summability.* The method used by Fejér, after suitable modifications, proved effective in deriving the sharpest result of the present paper, embodied in Theorems 9 and 10 below, where necessary and sufficient conditions for the uniform boundedness or uniform convergence of the sequence of partial sums $\{s_n(x)\}$ are obtained in terms of the "one-sided" uniform oscillation (from below) of the cosine and sine components of the Fourier series of $f(x)$.

2. The present section is devoted to a proof of the two following propositions.

THEOREM 1. *Let conditions* (1.1), (1.7), *and* (1.8) *be satisfied and let the series* $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$ *be slowly divergent in the sense of Definition* 1′. *Then the partial sums* (1.3) *of the Fourier series of* $f(x)$ *are uniformly bounded and the upper bound of* $|s_n(x)|$ *can be expressed in terms of L and of the characteristic data of the slow divergence of* $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$.

THEOREM 2. *If* $f(x)$ *is continuous in* $(-\pi, \pi)$ *and if conditions* (1.7) *and* (1.8) *are satisfied with* $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$ *slowly divergent in the sense of Definition* 1″, *then the Fourier series* (1.2) *of* $f(x)$ *converges to* $f(x)$ *uniformly for all* x.

* As yet unpublished; cf. a reference in the paper by M. Fekete and C. E. Winn, *On the connection between the limits of oscillation of a sequence and its Cesàro and Riesz means*, Proceedings of the London Mathematical Society, (2), vol. 35 (1933), pp. 488–513, especially p. 490. Since receiving the proof sheets, I have noticed that conditions equivalent to those required in Definitions 1′, 1″, 2′, 2″ occurred also in investigations of Landau and Schnee. Cf. Schnee's paper in the Proceedings of the London Mathematical Society, vol. 23 (1924), pp. 172–184.

The proof of these propositions is based on the following

LEMMA. *If, under conditions* (1.1), (1.7), *and* (1.8), *the series* $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$ *satisfy the conditions*

$$\sum_{\nu=k}^{k+q} \alpha_\nu \leqq A, \tag{2.1}$$

$$\text{for } k \geqq N \geqq 1,\ 0 \leqq q \leqq pk,\ p>0,$$

$$\sum_{\nu=k}^{k+q} \beta_\nu \leqq B, \tag{2.2}$$

then the partial sums (1.3) *admit of the estimate*

$$|s_n(x)| \leqq (5 + 2/p)L + 3(A + B), \text{ provided that } n \geqq N(1 + p). \tag{2.3}$$

To prove this lemma we start with the identity used by Paley*

$$\begin{aligned} s_n(x) = {} & \{n\sigma_n(x) - m\sigma_m(x)\}/(n - m) \\ & + \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu \cos \nu x + b_\nu \sin \nu x)/(n - m),\ 1 \leqq m < n. \end{aligned} \tag{2.4}$$

In view of (1.5) we have for the first term of the right-hand member of (2.4),

$$|n\sigma_n(x) - m\sigma_m(x)|/(n - m) \leqq (n + m)L/(n - m). \tag{2.5}$$

To evaluate the second term we consider, following Fejér and Paley,* the $(n-m)$th arithmetic means of the Fourier series of the functions $[f(x) + f(-x)] \cos nx$ and $[f(x) - f(-x)] \sin nx$, for $x=0$. Thus we obtain†

$$\left| \sum_{\nu=m+1}^{n} (\nu - m)a_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)a_\nu \right| \leqq 2L(n - m), \tag{2.6}$$

$$\left| \sum_{\nu=m+1}^{n} (\nu - m)b_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)b_\nu \right| \leqq 2L(n - m). \tag{2.7}$$

Consequently, in view of (1.7) and (1.8),

$$\begin{aligned} 0 & \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)(a_\nu + \alpha_\nu) \\ & \leqq (n - m)\left(2L + \sum_{\nu=m+1}^{2n-m} \alpha_\nu\right), \end{aligned} \tag{2.8}$$

$$0 \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) \leqq (n - m)\left(2L + \sum_{\nu=m+1}^{2n-m} \alpha_\nu\right). \tag{2.9}$$

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* Loc. cit., footnote ‡ on p. 238.

† Cf. our note referred to in footnote ‡ on p. 238 where more details are given.

On combining (2.9) with (1.7), (1.8), we get

$$
\begin{aligned}
&\left|\sum_{\nu=m+1}^{n}(\nu-m)a_\nu\cos\nu x\right| \\
(2.10)\qquad &\leqq \left|\sum_{\nu=m+1}^{n}(\nu-m)(a_\nu+\alpha_\nu)\cos\nu x\right| + \left|\sum_{\nu=m+1}^{n}(\nu-m)\alpha_\nu\cos\nu x\right| \\
&\leqq \sum_{\nu=m+1}^{n}(\nu-m)(a_\nu+\alpha_\nu) + \sum_{\nu=m+1}^{n}(\nu-m)\alpha_\nu \\
&\leqq (n-m)\left(2L+\sum_{\nu=m+1}^{2n-m}\alpha_\nu+\sum_{\nu=m+1}^{n}\alpha_\nu\right),
\end{aligned}
$$

and similarly

$$
(2.11)\qquad \left|\sum_{\nu=m+1}^{n}(\nu-m)b_\nu\sin\nu x\right| \leqq (n-m)\left(2L+\sum_{\nu=m+1}^{2n-m}\beta_\nu+\sum_{\nu=m+1}^{n}\beta_\nu\right).
$$

Now assume that the integers n and m satisfy the conditions

$$
(2.12)\quad n \geqq N(1+p),\ m = [n/(1+p)],\ \text{i.e.,}\ n/(1+p)-1 < m \leqq n/(1+p).
$$

Then

$$
(2.13)\qquad \begin{aligned} &m \geqq N \geqq 1,\ m \leqq n-1,\\ &n-m-1 < p(m+1) < p(n+1). \end{aligned}
$$

This enables us to apply (2.1) and (2.2) to estimate the sums of the right-hand members of (2.10), (2.11), with the result

$$
(2.14)\qquad \left|\sum_{\nu=m+1}^{n}(\nu-m)a_\nu\cos\nu x\right| \leqq (n-m)(2L+3A),
$$

$$
(2.15)\qquad \left|\sum_{\nu=m+1}^{n}(\nu-m)b_\nu\sin\nu x\right| \leqq (n-m)(2L+3B).
$$

Furthermore, for the values of m and n in question, (2.5) gives

$$
(2.16)\quad \left|n\sigma_n(x)-m\sigma_m(x)\right|/(n-m) \leqq (1+2/(n/m-1))L \leqq (1+2/p)L.
$$

Inequality (2.3) follows by an easy combination of (2.4), (2.16), (2.14), and (2.15).

The proof of Theorem 1 is now easily derived from the lemma above, whose conditions obviously are satisfied on the hypotheses of Theorem 1. If $n \geqq N(1+p)$ we use the estimate (2.3) directly. If $n < N(1+p)$ then by the Cauchy-Schwarz inequality and Parseval's theorem,

$$
\left|s_n(x)\right| \leqq (1+2n^{1/2})L \leqq \left\{1+(2N(1+p))^{1/2}\right\}L.
$$

Thus we obtain our final estimate

$$(2.17) \qquad |s_n(x)| \leqq \{5 + 2/p + 2(N(1+p))^{1/2}\}L + 3(A+B),$$

valid for all values of $n \geqq 0$.

We pass on to the proof of Theorem 2 and first observe that estimates (2.3) and (2.17), in view of the hypotheses of Theorem 2, will hold for an arbitrary choice of $A>0$ and $B>0$, provided N and p are suitably fixed as functions of A and B. Next we remark that the Fourier coefficients of

$$(2.18) \qquad \begin{aligned} f(x) - \sigma_n(x) \sim & \sum_{\nu=1}^{n} (\nu/n)(a_\nu \cos \nu x + b_\nu \sin \nu x) \\ & + \sum_{\nu=n+1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \end{aligned}$$

satisfy inequalities like (1.7) with the same α_ν's and β_ν's that accompanied the Fourier coefficients a_ν, b_ν of $f(x)$. Hence, on putting

$$M_n = \max_x |f(x) - \sigma_n(x)|, \quad m_n = \max_x |f(x) - s_n(x)|,$$

we derive the estimate corresponding to (2.17) for the nth partial sum of (2.18), namely

$$(2.19) \qquad \begin{aligned} |s_n(x) - \sigma_n(x)| &= \left| \sum_{\nu=1}^{n} (\nu/n)(a_\nu \cos \nu x + b_\nu \sin \nu x) \right| \\ &\leqq 3(A+B) + \{5 + 2/p + 2(N(1+p))^{1/2}\} M_n. \end{aligned}$$

Since

$$|f(x) - s_n(x)| \leqq |f(x) - \sigma_n(x)| + |s_n(x) - \sigma_n(x)|,$$

and, by Fejér's theorem, $M_n \to 0$ as $n \to \infty$, we see that

$$0 \leqq \limsup_{n\to\infty} m_n \leqq 3(A+B).$$

As $A>0$ and $B>0$ are arbitrary, we get finally $m_n \to 0$ as $n\to\infty$, which proves Theorem 2.

3. From Definitions 1′ and 1″ of slow divergence it is immediately seen that, if there exist at all sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the requirements of Theorems 1 or 2, then the particular sequences

$$\{\alpha_n^*\} = \{\tfrac{1}{2}(|a_n| - a_n)\}, \quad \{\beta_n^*\} = \{\tfrac{1}{2}(|b_n| - b_n)\}$$

will evidently satisfy these requirements. Consequently our Theorems 1 and 2 may be restated in the following form:

THEOREM 3. *If $f(x)$ satisfies condition* (1.1) *and if the series*

$$\sum_{n=1}^{\infty}(|a_n| - a_n), \qquad \sum_{n=1}^{\infty}(|b_n| - b_n) \tag{3.1}$$

are slowly divergent in the sense of Definition 1′, *then the partial sums $s_n(x)$ of the Fourier series of $f(x)$ are uniformly bounded.*

THEOREM 4. *If $f(x)$ is continuous and the series* (3.1) *are slowly divergent in the sense of Definition* 1″, *then the Fourier series of $f(x)$ converges to $f(x)$ uniformly.*

4. We now pass on to the generalizations of Theorems 1 and 2 mentioned in the Introduction. Using the notion of the slow oscillation of Definitions 2′, 2″ and of the uniform slow oscillation of the Remark following these definitions, we can enunciate the generalizations in questions as follows.

THEOREM 5. *Let $f(x)$ satisfy condition* (1.1) *and let its Fourier coefficients satisfy conditions* (1.7) *where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are such that the series $\sum_1^\infty \alpha_n \cos nx$ and $\sum_1^\infty \beta_n \sin nx$ are slowly oscillating* (*uniformly in x*) *in the sense of Definition* 2′. *Then the partial sums $s_n(x)$ of the Fourier series of $f(x)$ are uniformly bounded and the upper bound of $|s_n(x)|$ is expressible in terms of L and of the characteristic data of the slow oscillation of the series $\sum_1^\infty \alpha_n \cos nx$, $\sum_1^\infty \beta_n \sin nx$.*

THEOREM 6. *If $f(x)$ is continuous and if its Fourier coefficients satisfy conditions* (1.7) *where the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are such that the series $\sum_1^\infty \alpha_n \cos nx$ and $\sum_1^\infty \beta_n \sin nx$ are slowly oscillating* (*uniformly in x*) *in the sense of Definition* 2″, *then the Fourier series of $f(x)$ converges to $f(x)$ uniformly.*

The proof of these propositions is based on a lemma analogous to that of §2.

LEMMA. *If, under the assumptions of Theorem* 5, *the trigonometric series $\sum_1^\infty \alpha_n \cos nx$, $\sum_1^\infty \beta_n \sin nx$ satisfy the conditions*

$$\left|\sum_{\nu=k}^{k+q} \alpha_\nu \cos \nu x\right| \leqq A, \tag{4.1}$$

$$\left|\sum_{\nu=k}^{k+q} \beta_\nu \sin \nu x\right| \leqq B, \tag{4.2}$$

$$\text{for } k \geqq N \geqq 1,\ 0 \leqq q \leqq pk,$$

then the partial sums $s_n(x)$ admit of the estimate

$$|s_n(x)| \leqq (5 + 2/p)L + 6(A + B), \quad \text{provided that } n \geqq N(1 + p). \tag{4.3}$$

Using the notation and proceeding in the same fashion as in the proof of the lemma of §2, we now have, instead of (2.9),

$$
\begin{aligned}
(4.4)\qquad & 0 \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) \\
& \leqq 2(n-m)L + \left| \sum_{\nu=m+1}^{n} (\nu - m)\alpha_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)\alpha_\nu \right|.
\end{aligned}
$$

In view of (2.13) conditions (4.1) yield

$$
\left| \sum_{\nu=m+1}^{n} \alpha_\nu \cos \nu x \right| \leqq A, \qquad \left| \sum_{\nu=n+1}^{2n-m} \alpha_\nu \cos \nu x \right| \leqq A,
$$

whence it follows that

$$
(4.5)\qquad \left| \sum_{\nu=m+1}^{2n-m} \alpha_\nu \cos \nu x \right| \leqq 2A.
$$

On applying estimate (2.6) to the trigonometric polynomial of the left-hand member of (4.5) we have

$$
(4.6)\qquad \left| \sum_{\nu=m+1}^{n} (\nu - m)\alpha_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)\alpha_\nu \right| \leqq 4A(n-m).
$$

Being combined with (4.4) this gives

$$
(4.7)\qquad 0 \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) \leqq 2(n-m)(L + 2A).
$$

Hence

$$
\begin{aligned}
& \left| \sum_{\nu=m+1}^{n} (\nu - m) a_\nu \cos \nu x \right| \\
(4.8)\qquad & \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) + \left| \sum_{\nu=m+1}^{n} (\nu - m)\alpha_\nu \cos \nu x \right| \\
& \leqq 2(n-m)(L + 2A) + \left| \sum_{\nu=m+1}^{n} (\nu - m)\alpha_\nu \cos \nu x \right|.
\end{aligned}
$$

We now consider Paley's identity (2.4) with $f(x)$ replaced by the trigonometric polynomial

$$
t(x) = \sum_{\nu=m+1}^{n} \alpha_\nu \cos \nu x.
$$

If we denote by $\tau_\nu(x)$ the νth arithmetic mean associated with $t(x)$, we have

$$\frac{1}{n-m}\sum_{\nu=m+1}^{n}(\nu-m)\alpha_\nu\cos\nu x = t(x) - (n\tau_n(x) - m\tau_m(x))/(n-m).$$

In view of (2.13) and (4.1),

$$|n\tau_n(x) - m\tau_m(x)| = \left|\sum_{q=0}^{n-m-1}\sum_{\nu=m+1}^{m+1+q}\alpha_\nu\cos\nu x\right| \leqq (n-m)A.$$

Since also $|t(x)| \leqq A$, we have

$$\left|\sum_{\nu=m+1}^{n}(\nu-m)a_\nu\cos\nu x\right| \leqq 2(n-m)(L+3A), \tag{4.9}$$

and similarly

$$\left|\sum_{\nu=m+1}^{n}(\nu-m)b_\nu\sin\nu x\right| \leqq 2(n-m)(L+3B). \tag{4.10}$$

From this point on the argument proceeds in precisely the same fashion as in §2, and the proof of our lemma is complete. Theorem 5 now follows from the above lemma in precisely the same way as Theorem 1 was derived from the lemma of §2. In the present case we obtain the estimate

$$|s_n(x)| \leqq \{5 + 2/p + 2(N(1+p))^{1/2}\}L + 6(A+B), \quad n \geqq 0. \tag{4.11}$$

As to Theorem 6, it will be also proved on the basis of our lemma, but the method used in §2 to prove Theorem 2 can not be applied here and we have to resort to the original argument of Paley, which he used in discussing the uniform convergence of a Fourier series with positive coefficients.* Let $t_r(x; n)$ denote the partial sums of the Fourier series of

$$\begin{aligned} f(x) - \sigma_n(x) \sim \sum_{\nu=1}^{n}(\nu/n)(a_\nu\cos\nu x + b_\nu\sin\nu x) \\ + \sum_{\nu=n+1}^{\infty}(a_\nu\cos\nu x + b_\nu\sin\nu x). \end{aligned} \tag{4.12}$$

Under the hypotheses of Theorem 6 the series $\sum_1^\infty \alpha_\nu \cos \nu x$, $\sum_1^\infty \beta_\nu \sin \nu x$ are slowly oscillating in the sense of Definition 2''. It follows that the series

$$\sum_{\nu=1}^{\infty}\gamma_\nu^{(n)}\cos\nu x \equiv \sum_{\nu=1}^{n}(\nu/n)\alpha_\nu\cos\nu x + \sum_{\nu=n+1}^{\infty}\alpha_\nu\cos\nu x,$$

$$\sum_{\nu=1}^{\infty}\delta_\nu^{(n)}\sin\nu x \equiv \sum_{\nu=1}^{n}(\nu/n)\beta_\nu\sin\nu x + \sum_{\nu=n+1}^{\infty}\beta_\nu\sin\nu x$$

* Loc. cit. in footnote † on p. 237.

are also slowly oscillating in the sense of Definition 2′′ and that the constants $\gamma_\nu^{(n)}$, $\delta_\nu^{(n)}$ play the same part relative to the series (4.12) as α_ν, β_ν do relative to the Fourier series of $f(x)$. Indeed, for an arbitrary choice of $A>0, B>0$, there exist an integer $N=N(A, B)$ and a positive $p=p(A, B)$ such that

$$(4.13)\qquad \left|\sum_{\nu=k}^{k+q} \gamma_\nu^{(n)} \cos \nu x\right| \leqq A,$$

$$\text{for } k \geqq R = \max(n, N),\ 0 \leqq q \leqq pk.$$

$$(4.14)\qquad \left|\sum_{\nu=k}^{k+q} \delta_\nu^{(n)} \sin \nu x\right| \leqq B,$$

On applying the lemma above to the series (4.12) we conclude

$$(4.15)\qquad |t_r(x; n)| \leqq (5 + 2/p)M_n + 6(A + B), \quad r \geqq (1 + p)R,$$

where, as before,

$$M_n = \max_x |f(x) - \sigma_n(x)|.$$

Since

$$t_r(x; n) = \sum_{\nu=1}^{n} (\nu/n)(a_\nu \cos \nu x + \beta_\nu \sin \nu x) + \sum_{\nu=n+1}^{r} (a_\nu \cos \nu x + \beta_\nu \sin \nu x), \qquad r > n \geqq 1,$$

we have

$$|s_{r_2}(x) - s_{r_1}(x)| \leqq (10 + 4/p)M_n + 12(A + B), \quad r_2 > r_1 \geqq (1 + p)R.$$

Now, given any $\epsilon>0$, set $48A=48B=\epsilon$, which fixes also p and N. Choose n_0 so that $(10+4/p)M_n \leqq \epsilon/2$ when $n \geqq n_0$. Then $|s_{r_2}(x)-s_{r_1}(x)| \leqq \epsilon$, provided $r_2>r_1 \geqq (1+p)R_0$, $R_0=\max(n_0, N)$. Thus the Fourier series of $f(x)$ converges uniformly. The fact that its sum is $f(x)$ is implied by the classical theory of Fourier series. The proof of Theorem 6 is now complete.

5. It is clear that the uniform boundedness of the partial sums (1.3) implies the uniformly slow oscillation of the series

$$(5.1)\qquad a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

$$(5.2)\qquad \sum_{n=1}^{\infty} b_n \sin nx,$$

in the sense of Definition 2′, while the uniform convergence of the series (1.2) implies the uniformly slow oscillation of (5.1), (5.2) in the sense of Definition 2′′. This leads to the following two propositions, the proof of which is obvious in view of Theorems 5 and 6.

THEOREM 7. *The uniformly slow oscillation of the series* (5.1) *and* (5.2), *in the sense of Definition* $2'$, *is necessary and sufficient for the uniform boundedness of the partial sums of the Fourier series of a bounded* (*measurable and real-valued*) *function* $f(x)$.

THEOREM 8. *The uniformly slow oscillation of the series* (5.1) *and* (5.2), *in the sense of Definition* $2''$, *is necessary and sufficient for the uniform convergence of the Fourier series of a continuous function* $f(x)$.

6. Our final generalization of the results obtained heretofore is based on the notion of the one-sided slow oscillation (from above or from below) of a series.

DEFINITION $3'$. *A series* $\sum_1^\infty c_n$ *with real terms is said to be slowly oscillating from below if there exist two positive numbers* P *and* p *and a positive integer* N *such that*

$$\sum_{\nu=k}^{k+q} c_\nu \geqq -P, \quad \text{for } k \geqq N, \ 0 \leqq q \leqq pk. \tag{6.1}$$

DEFINITION $3''$. *A series* $\sum_1^\infty c_n$ *with real terms is said to be slowly oscillating from below if for an arbitrarily given positive* P *there exist a positive number* p *and a positive integer* N, *both depending on* P, *such that* (6.1) *holds.*

It is clear how these definitions should be modified in order to characterize slow oscillation from above, and also uniform slow oscillation, from below or from above.

We are in a position to state and prove

THEOREM 9. *A necessary and sufficient condition that the partial sums of the Fourier series of a bounded* (*measurable and real-valued*) *function* $f(x)$ *be uniformly bounded, is that the series* (5.1) *and* (5.2) *be slowly oscillating from below, in the sense of Definition* $3'$, *uniformly in* x.

THEOREM 10. *A necessary and sufficient condition that the Fourier series of a continuous function* $f(x)$ *converge uniformly to* $f(x)$ *is that the series* (5.1) *and* (5.2) *be slowly oscillating from below, in the sense of Definition* $3''$, *uniformly in* x.

The necessity of the conditions of Theorems 9 and 10 is obvious. The proof of sufficiency is based on the following

LEMMA. *Assume that the series* $\sum_1^\infty c_n$ *satisfies* (6.1) *and, in addition, that the arithmetic means of its partial sums* $s_n = c_1 + c_2 + \cdots + c_n$,

$$S_n = (s_1 + s_2 + \cdots + s_n)/n,$$

satisfy

$$|S_n| \leqq L < \infty, \qquad n \geqq 1. \tag{6.2}$$

Then the partial sums themselves satisfy

$$|s_n| \leqq (3 + 2/p)L + P, \text{ provided that } n \geqq N(1 + p). \tag{6.3}$$

To prove this lemma we use the identities

$$\begin{aligned}(n+m)S_{n+m} - nS_n - ms_n &= s_{n+1} + \cdots + s_{n+m} - ms_n \\ &= c_{n+1} + (c_{n+1} + c_{n+2}) + \cdots + (c_{n+1} + \cdots + c_{n+m}), \qquad m \geqq 1,\end{aligned} \tag{6.4}$$

$$\begin{aligned}ms_n - nS_n + (n-m)S_{n-m} &= ms_n - (s_{n-m+1} + \cdots + s_n) \\ &= c_n + (c_n + c_{n-1}) + \cdots + (c_n + \cdots + c_{n-m+2}), \qquad 1 \leqq m \leqq n.\end{aligned} \tag{6.5}$$

Now, assuming

$$n \geqq (1+p)N, \quad pn/(1+p) < m \leqq 1 + pn/(1+p), \tag{6.6}$$

we have

$$\begin{gathered} n > N, \quad n/m < 1 + 1/p, \\ 0 \leqq m - 1 \leqq pn/(1+p) < p(n+1), \\ n - m + 2 \geqq n/(1+p) + 1 \geqq N + 1 \geqq 2, \quad m - 2 < p(n - m + 2). \end{gathered}$$

We then may apply (6.1), which, being combined with (6.4), (6.5), gives

$$(n+m)S_{n+m} - nS_n - ms_n \geqq -mP, \tag{6.7}$$

$$ms_n - nS_n + (n-m)S_{n-m} \geqq -(m-1)P > -mP, \tag{6.8}$$

respectively. From (6.2), (6.7), (6.8) we derive

$$\begin{aligned} -P - (1 + 2n/m)L &< -P + (n/m)S_n - (n/m - 1)S_{n-m} \\ &\leqq s_n \leqq P + (1 + n/m)S_{n+m} - (n/m)S_n \leqq P + (1 + 2n/m)L. \end{aligned} \tag{6.9}$$

Since $n/m<1+1/p$ the desired inequality (6.3) follows at once.

The proof of Theorem 9 is now readily obtained from this lemma, since series (5.1) and (5.2), in view of (1.1) and Fejér's theorem, clearly satisfy all the requirements of the lemma. For $s_n(x)$ we get an estimate of the type (4.3). As to the proof of Theorem 10, it is easily obtained by an argument analogous to that used in proving Theorem 6, the details of which may be left to the reader.

THE EINSTEIN INSTITUTE OF MATHEMATICS OF THE HEBREW UNIVERSITY,
JERUSALEM, PALESTINE